VI. Curve $x^4 + y^4 = a^2(x^2 - y^2)$.

(i) By putting $x = 0$, $y = 0$, in the equation of the curve we find both the sides are satisfied. Hence the curve passes through the origin.

(ii) Since the equation of the curve contains only the even powers of x and y, therefore the curve is symmetrical about both the axes.

(iii) Putting $y = 0$ in the equation of the curve, we get

$$x^4 - a^2x^2 = 0 \Rightarrow x^2(x^2 - a^2) = 0 \therefore x = 0, \pm a,$$

That is, the curve cuts the x-axis at three points

$$(0, 0), (a, 0), (0, -a).$$

Again putting $x = 0$, we get $y^4 + a^2y^2 = 0$.

$\therefore y = 0$ i.e. the curve cuts the y-axis at the origin only.

(iv) By equating to zero the term of the lowest degree in the equation of the curve, we get $x^2 - y^2 = 0$ i.e. $y = \pm x$. Therefore the equations of the tangents at the origin will be

$$y = x, y = -x.$$

(v) There is no asymptote of the curve.

(vi) Solving for y from the equation of the curve, we get

$$y^4 + a^2y^2 = a^2x^2 - x^4 = x^2(a^2 - x^2).$$

If we give to x values $x > a$ or $x < -a$, then $a^2 - x^2 = -ve$ (say $-k$), so that $y^4 + a^2y^2 + k = 0$. Since there is no change of sign according to Descarte rule, therefore the roots of y are neither positive nor negative and hence y is imaginary. That is, there is no portion of the curve on the R.H.S. of $x = a$ or on the L.H.S. of $x = -a$.

(vii) Differentiating the equation of the curve w.r.t. x, we get

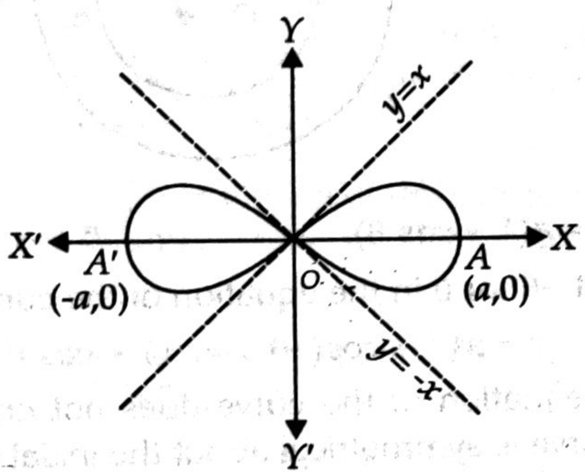
$$4x^3 + 4y^3\frac{dy}{dx} = a^2\left(2x - 2y\frac{dy}{dx}\right)$$

$$\Rightarrow \quad \frac{dy}{dx}(4y^3 + 2a^2y) = 2a^2x - 4x^3$$

$$\therefore \quad \frac{dy}{dx} = \frac{2a^2x - 4x^3}{4y^3 + 2a^2y}.$$

$\therefore$ At the point $(a, 0)$, $\frac{dy}{dx} = \infty$ and also at $(-a, 0)$ we have $\frac{dy}{dx} = \infty$ i.e. at the points $(a\ 0)$ and $(-a, 0)$, the tangent is perpendicular to the x-axis.

Hence the graph of the curve will be as follows :



VII. Curve $r = ae^{m\theta}$ (Equiangular spiral)

(i) r is always positive and increases with θ and when $\theta \to \infty$, then $r \to \infty$.

(ii) r is not zero for any finite value of θ and hence pole is not situated on the curve.

(iii) If $\theta = 0$, then $r = a$;

If $\theta = \frac{\pi}{2}$, then $r = ae^{\frac{m\pi}{2}}$

If $\theta = \pi$, then $r = ae^{m\pi}$ ....

Thus by giving to θ values greater than 0, r becomes increasingly greater than a.

(vi) From the equation $r = ae^{m\theta}$ of the curve, we have

$$\frac{dr}{d\theta} = ae^{m\theta} \cdot m = mr$$

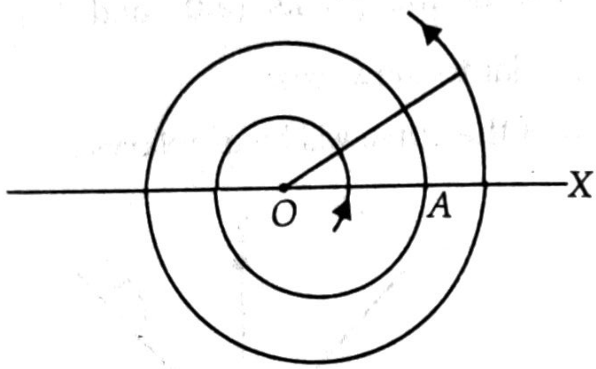
$$\therefore \qquad \tan\phi = \frac{r d\theta}{dr} = \frac{r}{mr} = \frac{1}{m}$$

$$\therefore \qquad \phi = \tan^{-1}\left(\frac{1}{m}\right) = \text{constant.}$$

Hence at any point the angle between the radius vector and tangent is constant.

This is the reason why this curve is called *equiangular spiral* and its *shape* is just like a spiral.

Hence the graph of the curve will be like this :



VIII. Cardiode $r = a(1 + \cos\theta)$.

(i) If we put $-\theta$ for θ in the equation of the curve, we find that

$$r = a\{1 + \cos(-\theta)\} = a(1 + \cos\theta)$$

i.e. the equation of the curve does not change. Therefore the given curve is symmetrical about the initial line.

(ii) $r = 0$ when $1 + \cos\theta = 0$ i.e. $\cos\theta = -1 \therefore \theta = \pi$. Hence the curve passes through the origin and the equation of the tangent at the pole is $\theta = \pi$ i.e. the initial line.

(iii) Now we plot some of the points on the curve.

When $\theta = 0$, then $r = 2a$;

when $\theta = \frac{\pi}{3}$, then $r = \frac{3}{2}a$;

when $\theta = \frac{\pi}{2}$, then $r = a$;

when $\theta = \pi$, then $r = 0$.

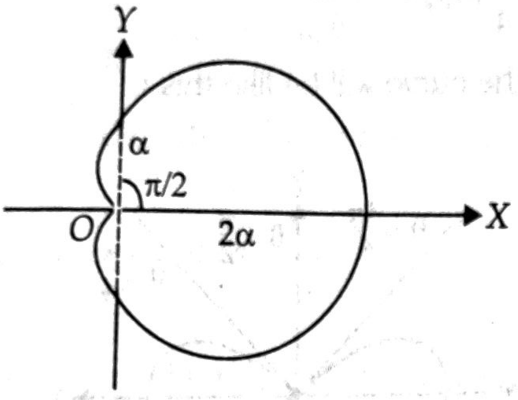
(iv) From the given equation $r = a(1 + \cos\theta)$, we have

$$\frac{dr}{d\theta} = -a\sin\theta.$$

That is, when the value of θ increases from 0 to π, then the value or r decreases and, as has been stated in (iii), decreases from $2a$ to 0.

Again, since the given curve is symmetrical about the initial line, therefore when the value of θ increases from π to 2π, then the value of r increases from 0 to $2a$.

Hence the graph of the curve will be like this :



IX. Curve $r^2 = a^2 \cos 2\theta$ (Lemniscate of Bernaulli)

(i) (a) If we put $-\theta$ for θ in the equation of the curve, then the equation of the curve does not change. Therefore the curve is symmetrical about the initial line.

(b) If we put $-r$ for r in the equation of the curve, the equation of the curve does not change. Therefore the curve is symmetrical about the pole.

(c) If put put $\pi - \theta$ for θ in the equation of the curve, then

$$r^2 = a^2 \cos 2(\pi - \theta) = a^2 \cos 2\theta$$

i.e. the equation of the curve does not change. Therefore the curve is symmetrical about the line $\theta = \frac{\pi}{2}$.

(ii) $r = 0$ when $\cos 2\theta = 0 \Rightarrow 2\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$

$\therefore \quad \theta = \frac{\pi}{4}, \frac{3\pi}{4}$.

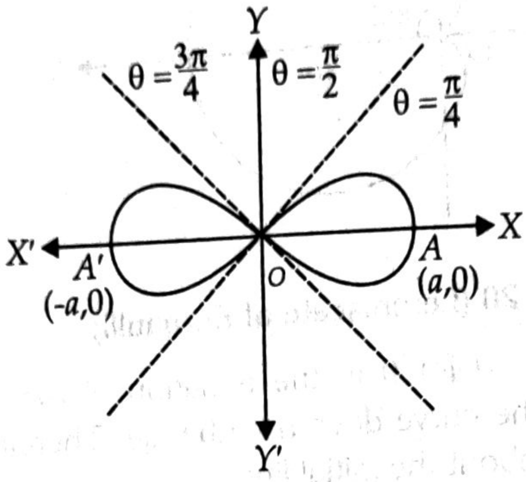
Hence the curve passes through the pole and the equations of the tangents at the pole are $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$.

(iii) $\theta = 0 \Rightarrow r^2 = a^2 \therefore r = \pm a$. Hence the points $(a, 0), (-a, 0)$ are situated on the curve.

(iv) When the value of θ increases from 0 to $\frac{\pi}{4}$, the value of r^2 decreases from a^2 to 0. Again, when the value of θ increases from $\frac{3\pi}{4}$ to π, the value of r^2 increases from 0 to a^2. But when the value of θ lies between $\frac{\pi}{4}$ and $\frac{3\pi}{4}$, then $r^2 = -ve$.

$\therefore$ r becomes imaginary. That is, no portion of the curve lies between $\theta = \frac{\pi}{4}$ and $\theta = \frac{3\pi}{4}$.

Hence the graph of the curve will be like this :



NOTE : Changing the equation of the curve in cartestian form, we get

$$r^2 = a^2(\cos^2\theta - \sin^2\theta)$$

$$\Rightarrow \quad r^4 = a^2(r^2\cos^2\theta - r^2\sin^2\theta)$$

But $x = r\cos\theta$ and $y = r\sin\theta$.

$$\therefore \quad x^2 + y^2 = r^2.$$

Hence $\quad (x^2 + y^2)^2 = a^2(x^2 - y^2)$.

Then its graph can be drawn like a cartesian curve.

X. Curve $r = a\sin 3\theta$

Soln. : (i) If we put $\pi - \theta$ for θ in the equation of the curve, we find that

$$r = a\sin 3(\pi - \theta)$$
$$= a\sin(3\pi - 3\theta) = a\sin(2\pi + \pi - 3\theta)$$
$$= a\sin(\pi - 3\theta) = a\sin 3\theta.$$

That is, the equation of the curve does not change. Therefore the given curve is symmetrical about the line $\theta = \frac{\pi}{2}$.

(ii) $r = 0$ when $\sin 3\theta = 0 \Rightarrow 3\theta = n\pi \;\therefore\; \theta = \frac{n\pi}{3}$

i.e. $r = 0$ when $\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \ldots$

Hence the curve passes through the pole and the equations of the tangents at the pole are $\theta = 0$, $\theta = \frac{\pi}{3}$, $\theta = \frac{2\pi}{3}$;

and when $\theta = \frac{\pi}{6}$, $r = a$; when $\theta = \frac{\pi}{2}$, $r = -a$; when $\theta = \frac{5\pi}{6}$, $r = a$.

(iii) We see that

(a) If θ increases from 0 to $\frac{\pi}{6}$, then the value of r increases from 0 to a;

if θ increases from $\frac{\pi}{6}$ to $\frac{\pi}{3}$, then the value of r decreases from a to 0.

Thus there is a loop between $\theta = 0$ and $\theta = \frac{\pi}{3}$.

(b) If θ increases from $\frac{\pi}{3}$ to $\frac{\pi}{2}$, then the value of r becomes $-a$ from 0;

if θ increases from $\frac{\pi}{2}$ to $\frac{2\pi}{3}$, then the value of r becomes 0 from $-a$.

Thus there is another loop between $\theta = \frac{\pi}{3}$ and $\theta = \frac{2\pi}{3}$.

(c) If θ increases from $\frac{2\pi}{3}$ to $\frac{5\pi}{6}$, then the value of r becomes a from 0;

if θ increases from $\frac{5\pi}{6}$ to π, then the value of r becomes 0 from a.

Thus there is a third loop between $\theta = \frac{2\pi}{3}$ and $\theta = \pi$.

(iv) From the given curve,

$$\frac{dr}{d\theta} = 3a \cos 3\theta.$$

$\therefore$ $$\tan \phi = r\frac{d\theta}{dr} = \frac{a \sin 3\theta}{3a \cos 3\theta} = \frac{1}{3}\tan 3\theta$$

$\therefore$ $$\phi = 0 \Rightarrow \tan 3\theta = 0 \Rightarrow 3\theta = n\pi,$$

i.e. $$\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi.$$

and $$\phi = \frac{\pi}{2} \Rightarrow \cot 3\theta = 0 \Rightarrow \cos 3\theta = 0$$

$\Rightarrow$ $$3\theta = (2n+1)\frac{\pi}{2} \quad \therefore \quad \theta = (2n+1)\frac{\pi}{6}$$

i.e. $$\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}.$$

Hence the graph of the curve will be like this :